Chapter 2

Limits and Derivatives

Chapter 2 introduces the fundamental idea of calculating **limits** in order to calculate the **tangent slope** of a curve, the **instantaneous velocity** of a moving object, the instantaneous rate of growth of a population, and other **rates of change**.

The second key concept is **continuity** of a function: the nice property of many but not all common functions that makes limits easy to compute.

Finally, we introduce the idea of the **derivative** of a function; the common mathematical idea behind tangent slopes, velocities and other rates of change.

By the end of this chapter, you should be able to answer all the **Concept Check Questions** except 5(b) and all of the **True-False Quiz Questions** except 14 in the Chapter 2 Review.

2.1 The Tangent and Velocity Problems

One of the beauties of mathematics is that often, several problems that seem to be quite different turn out to have very similar mathematical representations and solutions, so that there is s common way to solve them. Two such problems are:

- Making sense of the slope at a point on a curve.
- Finding the velocity of a moving object from knowing its position as a function of time.

The Tangent Problem

We know how to compute the slope of a straight line, and how this is related to, say, the slope of an inclined plank when the graph describes height as a function of horizontal position.

It is very useful to extend to this idea to calculating the slopes of curves. The slope can vary from point to point along a curve, so what we will calculate is the slope at each point of a curve. The geometrical idea is that near a point on a curve, the curve is very close to a certain straight line: the *tangent line* to that point.

Example 1, modified.
Find an equation of the tangent line to the parabola $y = x^2$ *at the point* $P(2, 4)$.

Note: we will work most exercises in class, so the solutions are usually not given in these notes; especially with examples like this that are done in the textbook!

Tangent lines at each point of a curve

We often want the tangent slope or tangent line at multiple points on the curve, or at all of them, and then it is more efficient to proceed as follows:

Example A: Tangent Lines to a Cubic Curve.
Find an equation of the tangent line to the cubic $y = x^3$ at the point $P(a, a^3)$ for any value a.

First we approximate the slope by the slope m_{PQ} of the secant line between this point P and a nearby point $Q(x, x^3)$ for x near a,

$$m_{PQ} = \frac{x^3 - a^3}{x - a}.$$

This should approach the tangent slope m as x approaches a [$x \to a$], and to see how m_{PQ} behaves then, it helps to **simplify first**. The numerator vanishes for $x = a$, so has a factor $x - a$, and when we divide out this factor, $x^3 - a^3 = (x - a)(x^2 + x \cdot a + a^2)$. This gives

$$m_{PQ} = \frac{x^3 - a^3}{x - a} = \frac{(x - a)(x^2 + x \cdot a + a^2)}{x - a} = x^2 + x \cdot a + a^2, \text{ for } x \neq a.$$

For x near a, this has values close to what we get by **substituting** a for x:
m_{PQ} gets close to $a^2 + a \cdot a + a^2 = 3a^2$.

Thus the tangent slope should be

$$m = \lim_{x \to a} m_{PQ} = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} (x^2 + xa + a^2) = 3a^2.$$

The **point-slope formula** then gives the tangent line at $P(a, a^3)$:

$$y = a^3 + 3a^2(x - a).$$

Note that a is some constant, only x is variable, so this is a line, not a more complicated polynomial. For example, at the point $Q(2, 8)$ given by $a = 2$, the tangent line is $y = 8 + 12(x - 2)$.

The Velocity Problem

One exercise is enough to reveal that the solution to this problem comes from the same calculations as seen above for computing tangent slopes:

Example 3 *Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450m above the ground. Find the velocity of the ball after 5 seconds.*

Exercise B. *For the previous example, find the velocity at any given time after the ball is dropped.*

Recycling ideas and methods of calculation

This section on the velocity problem is very short because in fact *we have already solved the velocity problem by solving the tangent problem.*

The ability to solve a few core problems, like the tangent problem, and then "recycle" the ideas and computational methods discovered for them when solving various other problems, is one key to the efficiency and utility of calculus.

The single most central idea discovered so far is finding **limits**: getting from various approximations to an exact answer, so we study that next.

Study Exercises 2*, 3, 5, 6*, 9; also review Examples 1 and 3.