Preview: Calculating Areas

For now we will look at just one mathematical problem, which was the starting point of the ideas of calculus more than 2000 years ago: computing the areas of regions in the plane. I suggest that you also read the sections on THE TANGENT PROBLEM and VELOCITY, which we will return to soon, in Chapter 2, and the SUMMARY on page 9.

With geometry, we can compute the area of any polygon, by dividing it into triangles and adding their areas. But how can we calculate the area of a regions with curved edges like a circle?
For example, how can we calculate π, the area of a unit circle?

This is perhaps the earliest of all calculus problems, and about 2500 years ago in Greece, it lead to the first use of the most distinctive strategy of calculus:

find an endless progression of ever more accurate approximations to the answer, so that as you move along this progression, you can get as close as you wish to the exact answer.

The idea with calculating the area of the unit circle is to start with what we know how to calculate, the areas of polygons, and approximate the area of a circle by regular polygons drawn inside the circle ("inscribed"), or surrounding it ("circumscribed").

As the number of sides increases, the inscribed polygon areas increase but stay less than than of the circle, while the circumscribed polygon areas decrease but stay more than than of the circle. The exact area is thus pinned between these low and high approximations.

The simplest example is squares: the inscribed square has diagonal 2, so side $\sqrt{2}$ and area 2; the circumscribed square has side 2, area 4, so $2 < \pi < 4$: a start, but very rough.

With some geometry and trigonometry, it can be shown that

- the inscribed polygon with n sides has area $I_n = \dfrac{n}{2} \sin \dfrac{360^o}{n}$, and
- the circumscribed polygon with n sides has area $C_n = \dfrac{n}{2} \tan \dfrac{360^o}{n}$

so that

- with 100 sides, $I_{100} = 3.139\ldots, C_{100} = 3.1457\ldots,$ so $3.139 < \pi < 3.146$.
- with 1000 sides, $I_{1000} = 3.14157\ldots, C_{1000} = 3.14163\ldots,$ so $3.14145 < \pi < 3.1417$.

and these results do appear to be closing in on $\pi = 3.1415926\ldots$.

Towards the end of the semester, in Chapter 5, we will use a similar method to find the area under the graph of a function, using rectangles instead of triangles.

The graph of the function $\sqrt{1-x^2}$ is a semi-circle with area $\pi/2$ under it, so we will be able to compute π that way too.