Chapter 1

Functions and Models (some important pre-calculus topics)

This chapter is mostly a review of pre-calculus, in conjunction with part of the preview assignments. In class we cover just the last two sections, on exponentials and logarithms, plus topics that come up from questions on the preview assignment.

The exercises for this chapter start with four online WebAssign diagnostic assignments. These correspond to the diagnostic tests near the front of the textbook, but submit your answers online even if you work them all out on paper first; partly to learn how to use WebAssign before we start on the graded assignments.

By the end of this Chapter, you should be able to answer all the Concept Check Questions and True-False Quiz questions from the Chapter 1 Review on pages 68 and 69..

1.4 Exponential Functions

A function like $f(x) = 2^x$ is called exponential because the argument x is the exponent in the formula. Exponential functions are the most basic and common *transcendental functions*, and are probably the most important functions in mathematics and science after polynomials. So this is the early encounter mentioned in the sub-title of the text.

We will see how exponential functions can be defined to have graphs that are continuous, unbroken curves with well defined slopes, rather being only a collection of separate points for integer values of x.

Non-negative integer powers of 2

With basic algebra, exponential functions are defined first for positive integer arguments, by formula

$$2^n = 2 \cdot 2 \cdot 2 \cdots 2, \text{ the product of } n \text{ factors } 2.$$

Then to satisfy the rule $2^{n+m} = 2^n \cdot 2^m$ for the case $m = 0$ requires

$$2^0 = 1$$

so all non-negative integers n are covered.

Negative integer powers of 2

For a negative integer n, $|n| = -n$ is positive, and to satisfy the rule $2^n \cdot 2^m = 2^{n+m}$ we must have $2^n.2^{-n} = 2^{n+(-n)} = 2^0 = 1$, and so dividing by $2^{-n} = 2^{|n|}$,

$$2^n = \frac{1}{2^{-n}} = \frac{1}{2^{|n|}} \quad \text{for } n \text{ a negative integer.}$$

Rational powers of 2

Next we can make sense of exponentials for rational exponents. To get the exponential 2^r for any rational number r start with exponent $1/q$, q a positive integer. To satisfy the rule $(2^a)^b = 2^{ab}$ requires $(2^{1/q})^q = 2^{q/q} = 2^1 = 2$, so taking the q-th root of both sides of this equation,

$$2^{1/q} = \sqrt[q]{2} \text{ (the } q\text{-th root of 2) for } q \text{ a positive integer.}$$

Finally, any rational number can be written as $r = p/q$ with p an integer, q a positive integer, and the same rule requires

$$2^{p/q} = ((2^p)^{1/q}) = \sqrt[q]{2^p}.$$

Irrational powers of 2 (so all power of 2)

The graph of 2^x for all rational x looks like a dense collection of dots along a curve which increases to the right. Can we fill in the gaps at irrational values of x and get an smooth, uninterrupted curve? For example, can we make sense of an irrational power like $2^{\sqrt{3}}$?

A number like $\sqrt{3} = 1.73205\ldots$ is approximated by a succession of decimal fractions 1, 1.7, 1.73, 1.732, 1.7320, 1.73205 and so on: it is the *limit* of this *sequence* of rational numbers. Raising 2 to each of these powers gives the following new sequence of numbers (everything rounded to five decimal places):

$$2^1 = 2 < 2^{1.7} = 3.24900 < 2^{1.73} = 3.31727 < 2^{1.732} = 3.32188 < 2^{1.73205} = 3.32200\ldots$$

All of these should be less that $2^{\sqrt{3}}$ since the values are increasing as the exponent increases and $\sqrt{3}$ is greater than each of these exponents. On the other hand if we round up the decimal approximations of $\sqrt{3}$, the exponentials should all be greater than $2^{\sqrt{3}}$:

$$2^2 = 4 > 2^{1.8} = 3.48220 > 2^{1.74} = 3.34035 > 2^{1.733} = 3.32418 > 2^{1.7321} = 3.32211 > 2^{1.73206} = 3.32202\ldots$$

So it appears that

$$2^{1.73205} = 3.32200 < 2^{\sqrt{3}} < 3.32202 = 2^{1.73206},$$

so that $2^{\sqrt{3}}$ rounded to four decimal places is 3.3220.

We could continue with either sequence to compute a value for $2^{\sqrt{3}}$ to as many decimal places as we wish.

In this way, we can make sense of, and compute, any power of 2, rational or irrational, so we have made sense of the exponential function $f(x) = 2^x$ for all real arguments x.

Irrational powers of any positive number

There is nothing special about the base 2 used above except that it is positive: we could do the same thing with any positive real number a, to compute the exponential function $f(x) = a^x$. The graphs for the different functions vary mostly in that they are increasing for $a > 1$, and increase faster for larger values of a, and are decreasing for $0 < a < 1$, decreasing faster for smaller values of a.

In the borderline case of $a = 1$, the graph is a constant: $1^x = 1$.

Rules for exponential functions

The familiar rules for exponentials still hold just as with with rational exponents: for a and b positive and any real numbers x and y,

- $a^{x+y} = a^x \cdot a^y$, and $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{x \cdot y}$
- $(a \cdot b)^x = a^x \cdot b^x$

Note: from now on, marginal notes like that at right will indicated examples in the textbook to study in conjunction with ideas just discussed in these notes. Examples 1 & 2

Applications of exponential functions

Read for yourself first textbook example os about population growth, which we will see that again later.

Exercise A. *The half-life of strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.* Example 3

a. If a sample of ^{90}Sr initially has a mass of 24mg, find an expression for the mass $m(t)$ that remains after t years.

b. Find the mass remaining after 40 years, correct to the nearest milligram.

c. Use a graphing device to graph $m(t)$ and use the graph to estimate the time required for the mass to be reduced to 5 mg.

The number e

Of all possible choice of the base a of an exponential function a^x, one is most convenient for mathematics because it makes the slope of the graph simplest: the number called e, with value approximately $e \approx 2.71828$.

The graphs of all exponential functions pass through the point $P(0,1)$ on the y-axis, but the bigger a is, the faster the function value grows as x increases, so the greater the slope is at this point. The slope is zero for $a = 1$, when the function is constant, and increases as a increases. Experimenting with a graphing calculator suggests that the slope is less than 1 for 2^x, but greater than 1 for 3^x.

So it seems that by increasing a to somewhere between 2 and 3, the slope will be 1 at $P(0,1)$, with the slope greater than 1 for greater values of a, less than 1 for lesser values. That is, there is one special value for the base that gives slope 1: this is the value called e.

We have already seen that e lies between 2 and 3, and with ever more careful computation of slopes we could calculate the more accurate value given above.

We will soon see that any other exponential function can be written in terms of e^x, and this is very convenient in calculus, making this particular exponential function so important that it is often called simply “the exponential function”. Examples 4 & 5

Study Exercises 1^*, 3^*, 9, 11^*, 15, 18^*, 17, 21^*, 24^*, 30^*, 34^*

1.5 Inverse Functions and Logarithms

In part (c) of Example A of Section 1.5, we knew that the value of the function $m(t)$ (mass of Strontium-90 remaining) was 5 mg, and wanted to know the corresponding value of its argument t (the time). More generally it would be useful to have a formula giving time t as a function of mass m, $t = g(m)$. A function like this that takes values "backwards" compared to the function is the *inverse* of that function.

For example, with $y = f(x) = x^3$, we get back from a y value to the corresponding x value by treating y as known and solving $y = x^3$ for the unknown x: this gives $x = \sqrt[3]{y}$. Thus the cube root function is the inverse of the cube function, and we write $x = f^{-1}(y) = \sqrt[3]{y}$. It is often convenient to go back to using the name x for the argument of this new function too, writing $f^{-1}(x) = \sqrt[3]{x}$.

The horizontal line test and one-to-one functions

Graphically, one gets the graph of the inverse function $x = f^{-1}(y)$ by flipping the graph of $y = f(x)$ along the diagonal line $y = x$. Since the role of the x and y values are swapped, the domain of the inverse is the range of the original function, and vice versa.

But this flipping does not always give the graph of a function. The graph of any function must pass the *vertical line test* that no vertical line intersects it more than once, and for the flipped graph to pass, the original graph must have no *horizontal* line intersects it more than once. This is the **horizontal line test**, and is exactly what is needed for a function to have an inverse. Algebraically, this means that no two different arguments x_1 and x_2 give the same value of the function:

Definition. *Function f is **one-to-one** if for any $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.*

Examples 1 & 2

The example $f(x) = x^3$ is one-to -one and passes the HLT, since the function is increasing and so passes through any horizontal line just once and never returns. In fact, any function that is always increasing passes the HLT (same if it is always decreasing).

On the other hand $f(x) = x^2$ fails the HLT: the horizontal line $y = 1$ intersects for both $x = 1$ and $x = -1$. In fact every line $y = a$ for positive a intersects at two x values, $\sqrt{a}$ and $-\sqrt{a}$.

Inverse functions

One-to-one functions are exactly the ones that have inverses:

Definition. *If function f is **one-to-one**, if has an inverse, denoted f^{-1}, and defined by $x = f^{-1}(y)$ being given by the solution x of the equation $f(x) = y$.*

If a function is not one-to-one, this equation has several solutions for the same y-value, so does not determine the value x so the inverse is not defined.

The domain of f^{-1} is the range of f [the "y-values" in the above equation], and the range of f^{-1} is the domain of f [the "x-values"].

But how do we reconcile $f(x) = x^2$ not being one-to-one with it having an inverse, the square root function? Be careful: we are talking about two different functions here, even though they are described using the same formula $y = x^2$! When we use the formula with domain all the real numbers, it is not one-to-one, and has no inverse, but when we change the domain to non-negative real numbers, that is a different function, with inverse $\sqrt{x}$.

The difference with this smaller domain is that the graph is only the right half of a parabola, which is increasing and so satisfies the HLT. Algebraically, for any given value y, the equation $x^2 = y$ has only one non-negative solution x. Example 3–5

Notation warning: inverses are not reciprocals!

Beware of a possible confusion:
$y = f^{-1}(x)$ [the **inverse** of function f applied to x] is not the same as
$y = [f(x)]^{-1} = \dfrac{1}{f(x)}$ [the **reciprocal** of $f(x)$].

Logarithmic functions

Does an exponential function $y = f(x) = a^x$ have an inverse?

For $a > 1$, the value of a^x increases as x increases: the graph is increasing, which is enough to pass the HLT and ensure existence of an inverse. For $0 < a < 1$, the graph is decreasing and so again passes the HLT, giving an inverse. (For $a = 1$, there is no inverse.)

This inverse should be familiar: the number x for which $a^x = y$ is called the *logarithm of y base a*, written $\log_a y$, so the inverse of the exponential function $f(x) = a^x$ is the *logarithmic function base a*, $f^{-1}(x) = \log_a x$. An exponential function for $a \neq 1$ is defined for all real numbers (its domain) and its values (range) are all positive numbers. Thus the logarithmic functions $\log_a$ have domain all the positive numbers, range all the reals: only positive numbers have logarithms.

Note: this simple domain and range for logarithms depends on exponential functions now being defined for all real arguments, not just all rational arguments.

Rules for logarithms

Logarithms satisfy the following rules, all following from the rules for exponentials in Section 1.5. For any positive number a except 1, and any positive numbers x and y,

- $\log_a(x \cdot y) = \log_a x + \log_a y$
- $\log_a(x/y) = \log_a x - \log_a y$
- $\log_a(x^p) = p \log_a x$ for any real power p.

Example 6

Natural logarithms

Since e^x is the most commonly used exponential function, its inverse $\log_e$ is the most important logarithmic function. It is called the *natural logarithm*, and has the special name $\ln$ (from the initials of "logarithm" and "natural"):

$$\ln x = \log_e x.$$

Examples 7–9

Our first use of the natural logarithm is to put any exponential function a^x in terms of e^x. Using the properties of exponentials and the fact that $e^{\ln a} = a$,

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}.$$

It can also be shown that

$$\log_a x = \frac{\ln x}{\ln a},$$

so we can also put all logarithmic functions in terms of natural logarithms. Thus we mostly need just one exponential function, e^x, and just one logarithmic one: its inverse, the **natural logarithm**.

Example 10

Study Exercises 3, 4*, 6*, 10*, 13, 17, 18*, 19, 22*, 25. We omit the final topic, INVERSE TRIGONOMETRIC FUNCTIONS, for now; instead we will review it when we encounter these functions in Section 3.5. Example 11