2.2 The Limit of a Function

**Definition** (Limit, informal version). For a function \( f \) and numbers \( a \) and \( L \), we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \) if we can force the value of \( f(x) \) to be as close to \( L \) as we wish by considering only values of \( x \) sufficiently close to \( a \), but not equal to \( a \). This is written as

\[
\lim_{x \to a} f(x) = L
\]

Note that the value of \( f(a) \) is irrelevant: \( f \) need not even be defined for \( x = a \).

**Exploring All Nearby \( x \) Values**

Examples 2 and 4 of the text show that it is important to consider all values of \( x \) near \( a \) when studying a limit as \( x \to a \), not just a selection.

**Example A.** Show that \( \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x^2 - 2}{x - 1} = 4 \).

We can simplify to \( f(x) = 2x + 2 \), valid for all \( x \neq 1 \). Then we measure how close two numbers are by the absolute value of their difference. For example, if \( x \) is within 0.001 of 1, \( |x - 1| < 0.001 \), and so \( |f(x) - 4| = |(2x + 2) - 4| = |2x - 2| = 2|x - 1| \) which is less than 0.002. When we look only at \( x \) values ever closer to 1, in that \( |x - 1| \) is ever smaller, \( |f(x) - 4| = 2|x - 1| \) is ever smaller: \( f(x) \) gets ever closer to 4. For example, the value \( f(x) \) is sure to be within a tiny \( 10^{-100} \) of 4 when we look at \( x \) values within 0.5 \( \times 10^{-100} \) of 1. So the limit is 4.

**Example B: A Function With a Jump** Consider the function \( f(x) \) given by

\[
f(x) = \begin{cases} 
2x^2 - 2, & x \neq 1 \\
1, & x = 1 
\end{cases}
\]

What is the limit of \( f(x) \) as \( x \) approaches 1?

Since the limit as \( x \to a \) is based on values of \( f(x) \) for all \( x \) near \( a \), but not equal to \( a \), only the formula \((2x^2 - 2)/(x - 1)\) matters! And since it is equal to \( 2x + 2 \) for all \( x \) values near 1, the value is near \( 2 \cdot 1 + 2 = 4 \) there, and the limit is 4: \( \lim_{x \to 1} f(x) = 4 \), not 1.

**Note well:** The limit of \( f(x) \) as \( x \) goes to \( a \) does not always equal the value \( f(a) \), even when \( f(a) \) makes sense!

The graph of this function has a jump at \( x = 1 \), but the limit calculation ignores this, and treats the function as if it were “uninterrupted” or “continuous” there.

**Example 6 p. 92: Another type of jump: the Heaviside function**

In the physical description of sudden changes, like turning on a power switch, the **Heaviside Function** is often useful:

\[
H(t) = 0 \text{ for } t < 0, \ H(t) = 1 \text{ for } t \geq 0
\]

For \( t \) near 0 and positive, \( H(t) \) is 1, suggesting a limit of 1.

But for \( t \) near 0 and negative, \( H(t) \) is 0, suggesting a limit of 0.

The limit cannot be both zero and one, so again this function has no limit as \( t \to 0 \), due to this jump from one value to another, which breaks the graph at this point.

**One-sided Limits**

In the example above, we see that \( H(t) \) has “no limit” as \( t \to 0 \), but it is useful also to describe what happens at times just before \( t = 0 \), and what happens at times just after \( t = 0 \): what happens to one side or the other of a point on the graph.
We want to note that “as \( t \) approaches 0 from the right (\( t > 0 \)), \( H(t) \) approaches 1.” We use the notation \( t \to 0^+ \), with a plus sign superscript indicating that only \( t \) values to the right are considered: the relevant \( t \) values are “0 plus something”. The value approached is the right-hand limit, or the limit from the right, with short-hand notation

\[
\lim_{t \to 0^+} H(t) = 1.
\]

Similarly the behavior for \( t \) near 0 and less than zero is called the left-hand limit and we use a minus sign superscript, because the \( t \) value is “0 minus something:”

\[
\lim_{t \to 0^-} H(t) = 0.
\]

Note well: “\( t \to 0^- \)” is different from “\( t \to -0 \)”, which would be a funny way of writing a normal “two-sided” limit. And \( t \to 1^- \) is very different than \( t \to -1 \); the former is about what happens for \( t \) just below 1; the latter is about what happens for \( t \) near –1.

The limit exists when both one sided limits exist and agree

Comparing definitions, the limit of a function as \( x \to a \) exists exactly when both one sided limits exist, and both give the same value. Sometimes, computing a limit one side at a time is easiest: in particular when the function is given by different formulas on the two sides.

Example 7

Infinite Limits

We have seen several ways that a function can fail to have a limit as \( x \to a \), and decided that sometimes, there is still something useful to say about how the function behaves for \( x \) near \( a \) (one-sided limits). Example 8, page 94 gives another case of that: trying to compute the limit of \( 1/x^2 \) as \( x \to 0 \).

This is not a normal limit with a real number as its value, as discussed above: we say that \( 1/x^2 \) has an infinite limit at \( x = 0 \).

Of course the values of \( f(x) \) could also go the opposite way, down to ever lower values with no lower bound. For example, we say that \( \lim_{x \to 1} (x - 1)^2 = -\infty \).

One-sided Infinite Limits

Finally, it is natural to combine the ideas of infinite limits and one-sided limits.

Exercise C. Describe how \( f(x) = \frac{1}{x^2 - 2} \) behaves for \( x \) near 2, for the two cases \( x > 2 \) and \( x < 2 \).

The values get large and positive on one side, large and negative on the other, so for \( x \) coming from the right, “the value approaches \( \infty \)”, while from the left, “the value approaches \(-\infty \)”.

Combining the above ideas and notation of one sided limits and infinite limits, we state this as

\[
\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty, \quad \lim_{x \to 2^+} \frac{1}{x - 2} = \infty.
\]

But the limits from the two sides are different, so

\[
\lim_{x \to 2} \frac{1}{x - 2} \text{ Does Not Exist (DNE)}.
\]

Examples 9,10

Study Exercises Do Exercises 1-4, 6*, 15, 18*, 19, 27, 28*; review all Examples.