Lecture 8 : The Topology of Neural Nets and Deep Learning

May 16, 2019

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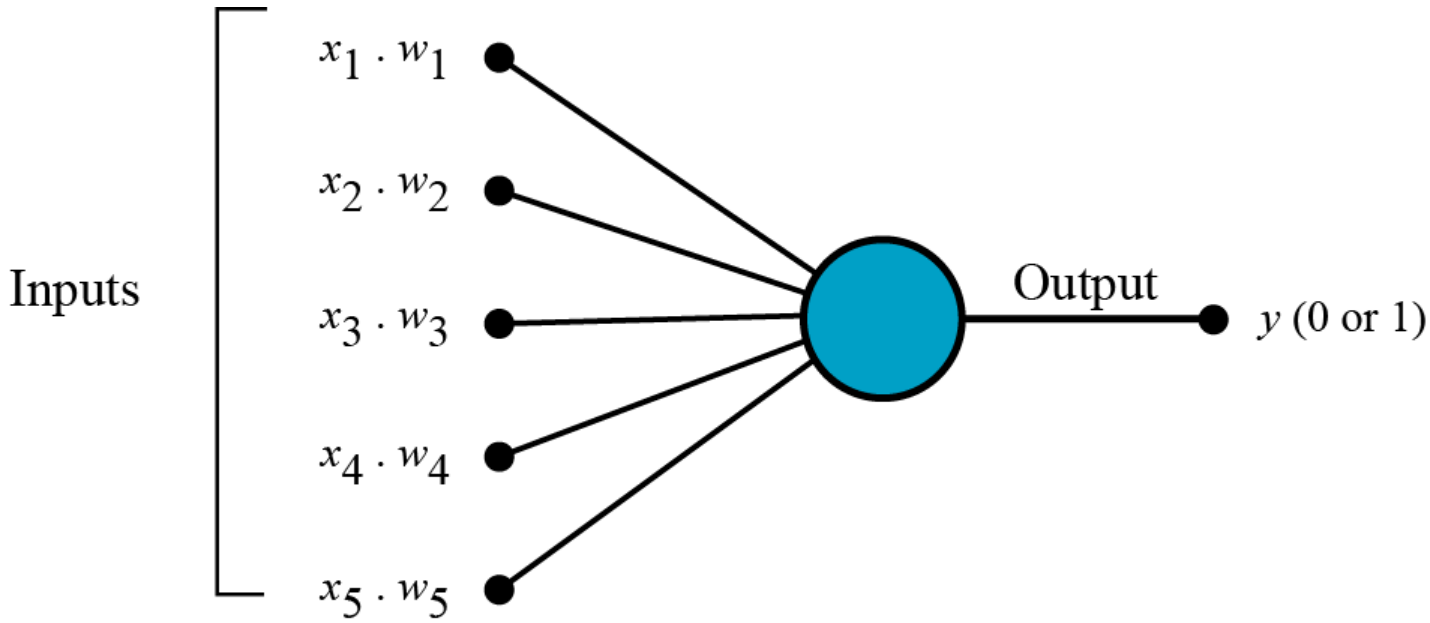
Perceptrons



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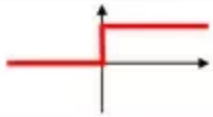
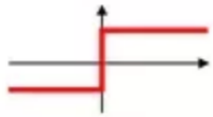
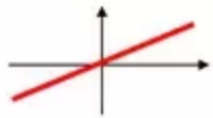
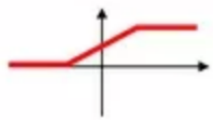
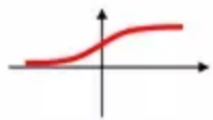
- Ultimately generalize to include continuous functions as well as Boolean ones
- Couple together to get networks of perceptrons

Activation Functions

Activation function	Equation	Example	1D Graph
Unit step (Heaviside)	$\phi(z) = \begin{cases} 0, & z < 0, \\ 0.5, & z = 0, \\ 1, & z > 0, \end{cases}$	Perceptron variant	
Sign (Signum)	$\phi(z) = \begin{cases} -1, & z < 0, \\ 0, & z = 0, \\ 1, & z > 0, \end{cases}$	Perceptron variant	
Linear	$\phi(z) = z$	Adaline, linear regression	
Piece-wise linear	$\phi(z) = \begin{cases} 1, & z \geq \frac{1}{2}, \\ z + \frac{1}{2}, & -\frac{1}{2} < z < \frac{1}{2}, \\ 0, & z \leq -\frac{1}{2}, \end{cases}$	Support vector machine	
Logistic (sigmoid)	$\phi(z) = \frac{1}{1 + e^{-z}}$	Logistic regression, Multi-layer NN	
Hyperbolic tangent	$\phi(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$	Multi-layer NN	

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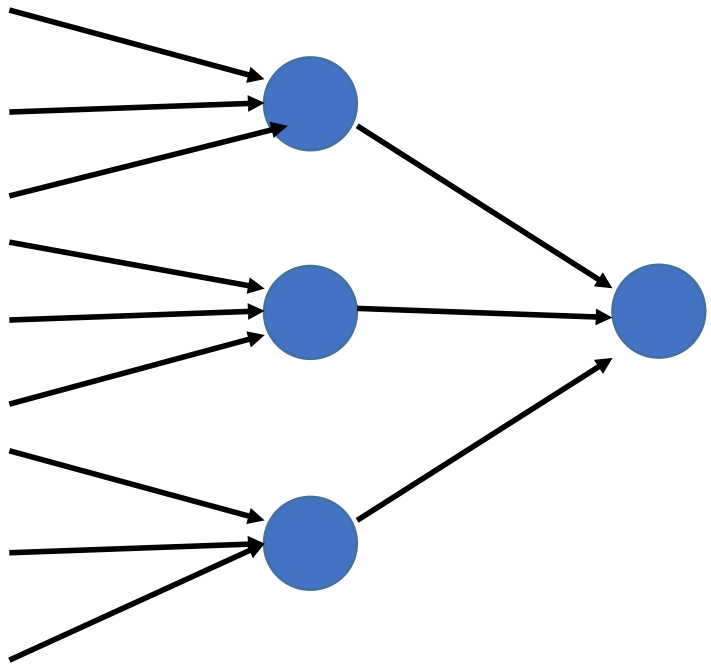

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- Different architectures for different types of data and different types of problems
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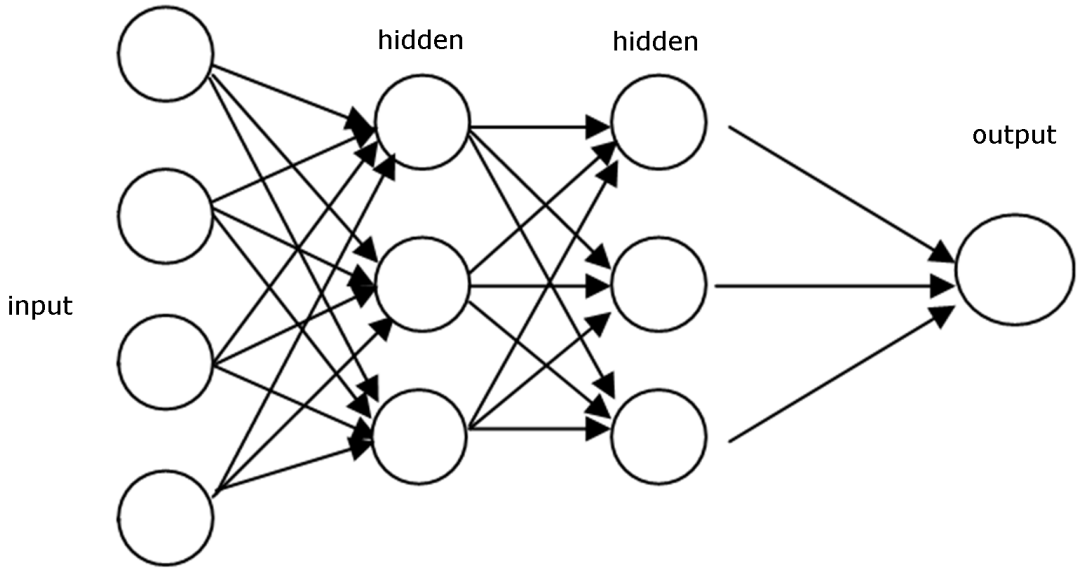
- *Many* variable optimization problem

Feed Forward Networks



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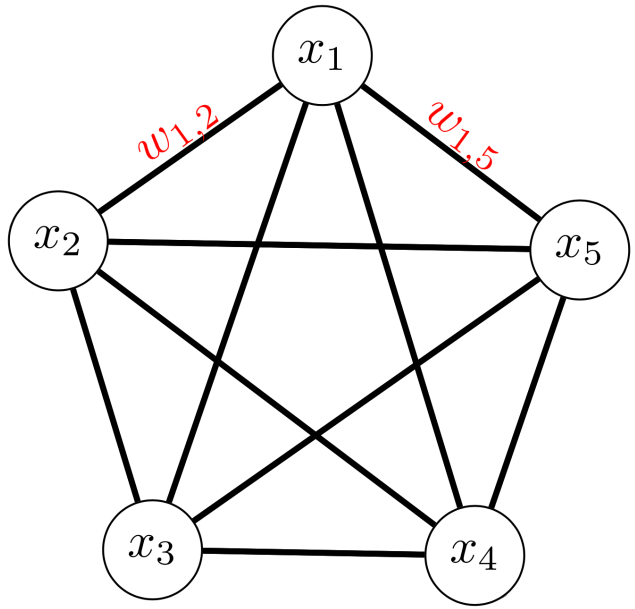
- One optimizes a fit of a function of this form to given function over the space of all possible weights

Hopfield Networks



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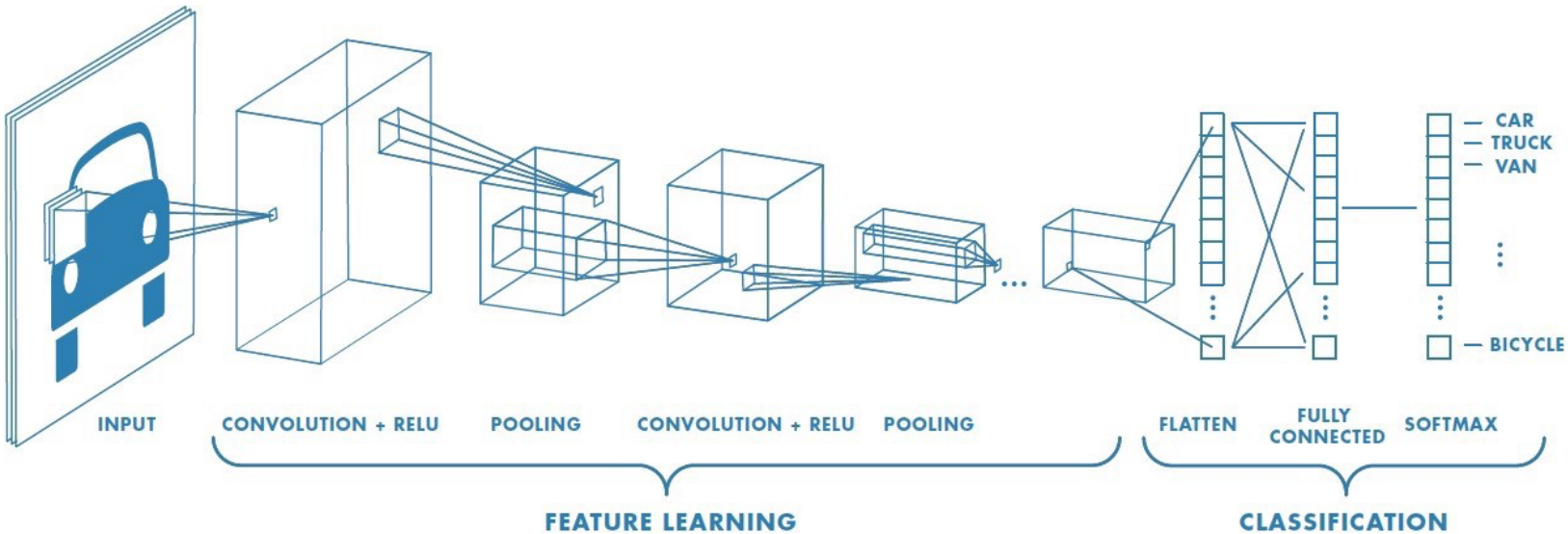
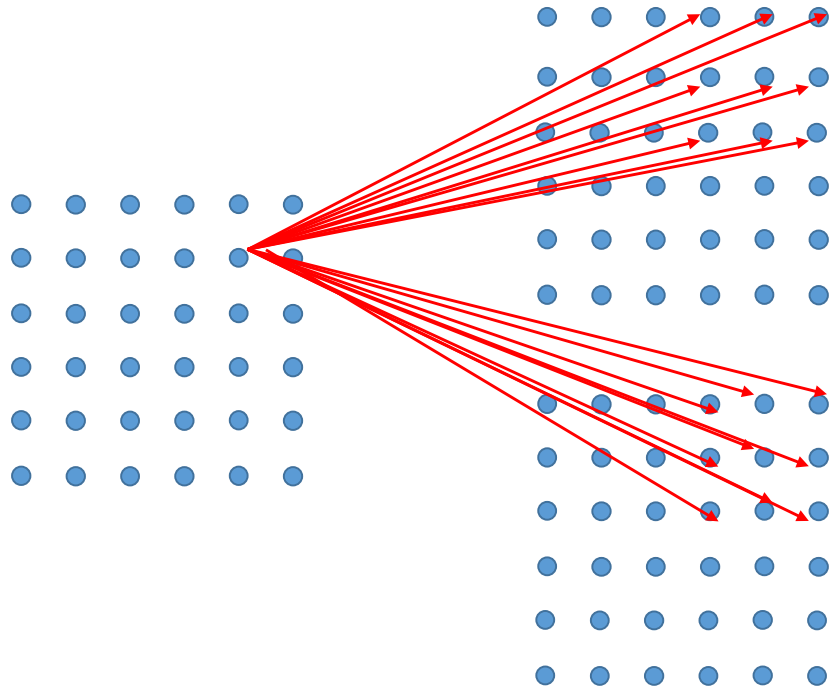
- The examples are local minima of an energy function
- A dynamical system is created which flows to the minima of the energy function
- An initial point which is close to one of the examples would naturally flow to the corresponding local minimum

Convolutional Neural Networks



Convolutional Neural Networks



Locality

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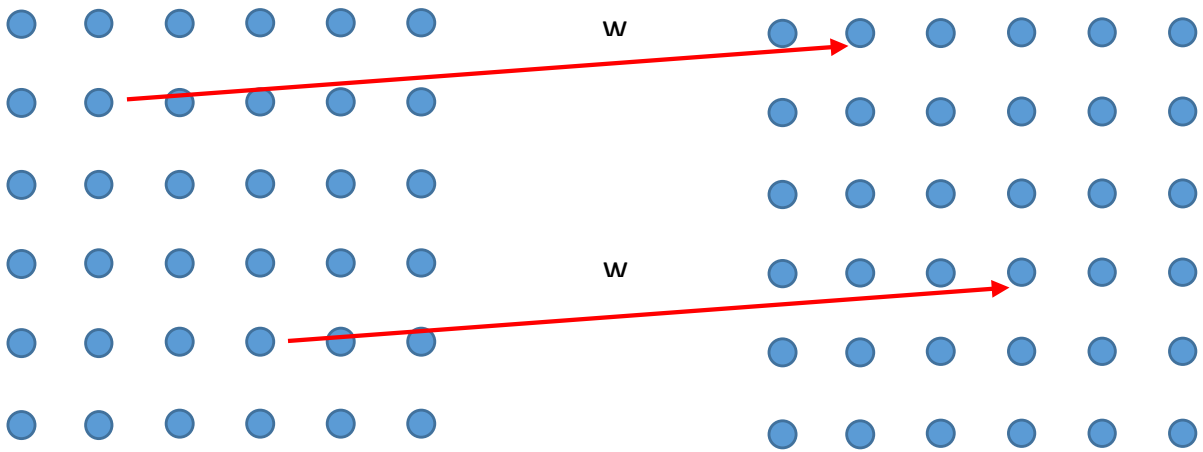
- Restriction means models are smaller and learning is faster than fully connected situation

Convolutional Neural Networks



Homogeneity or "Convolutionality"

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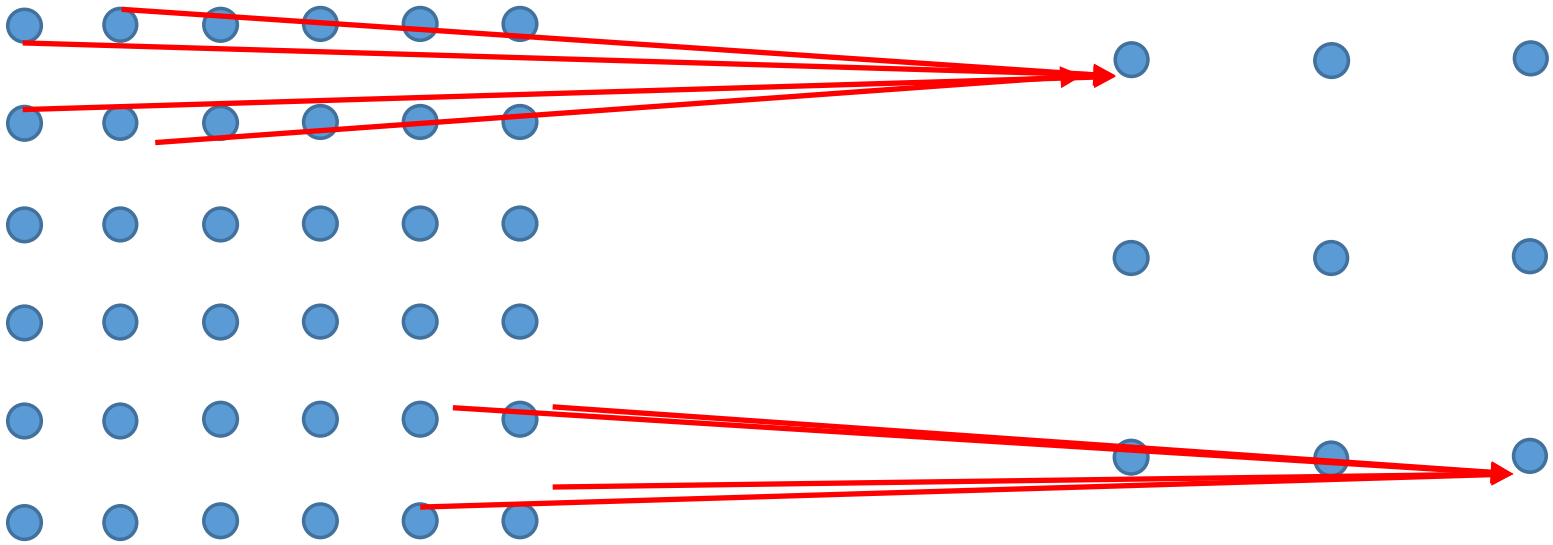
- Exploits symmetries in space of features
- Can exploit this idea for other feature geometries

Convolutional Neural Networks



Pooling

Convolutional Neural Networks

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- Mimics our understanding of how human visual pathway works

Problems with Deep Learning

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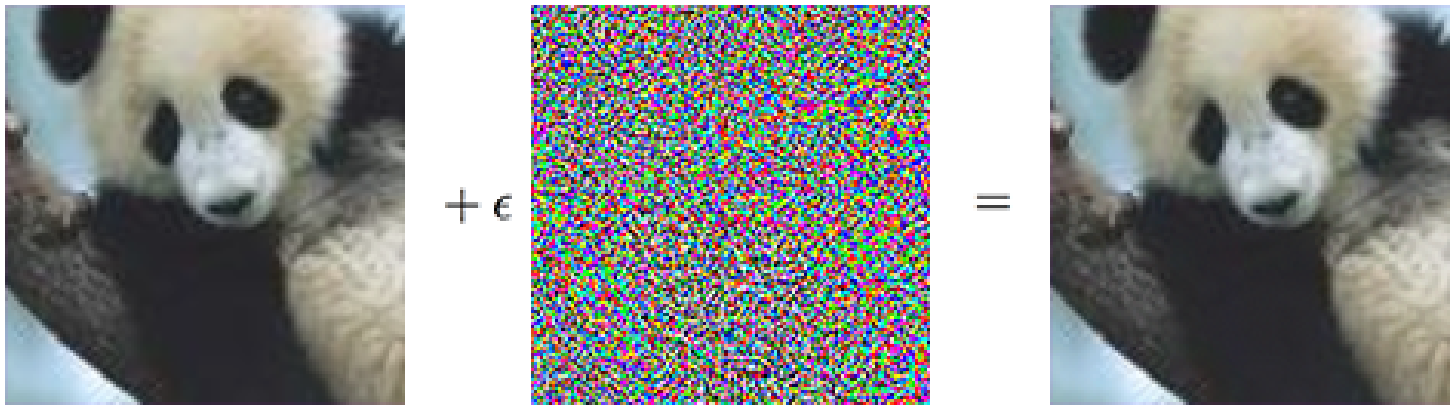
- Adversarial Examples

Adversarial Examples



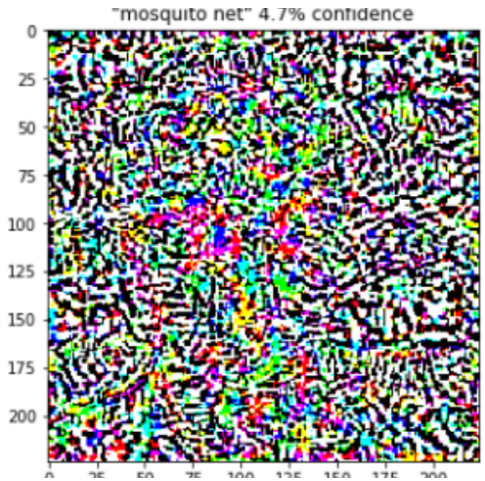
"panda"

57.7% confidence

"gibbon"

99.3% confidence

Adversarial Examples



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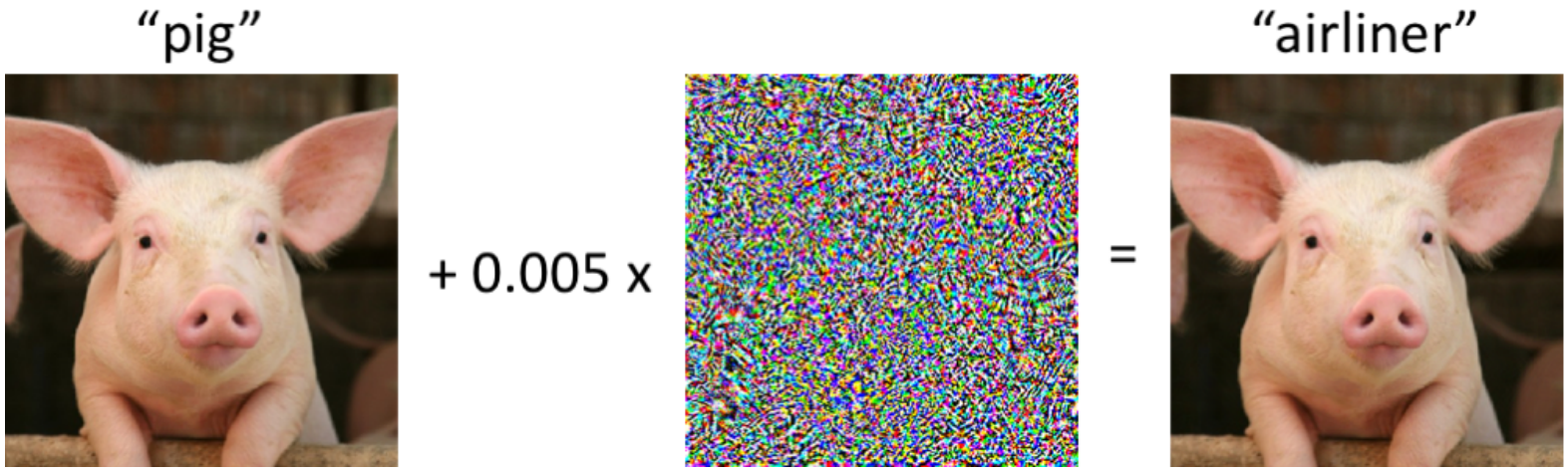
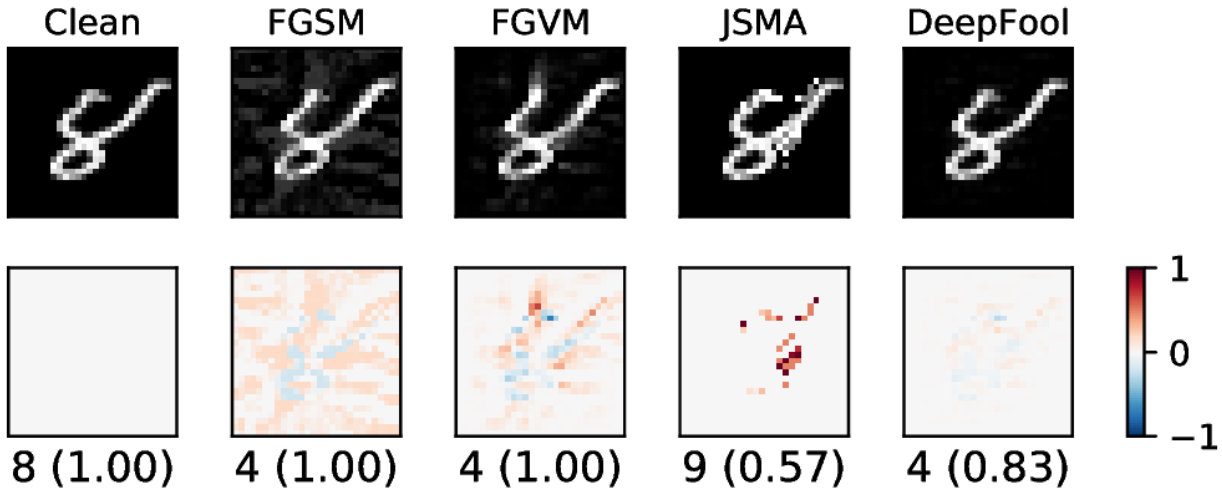
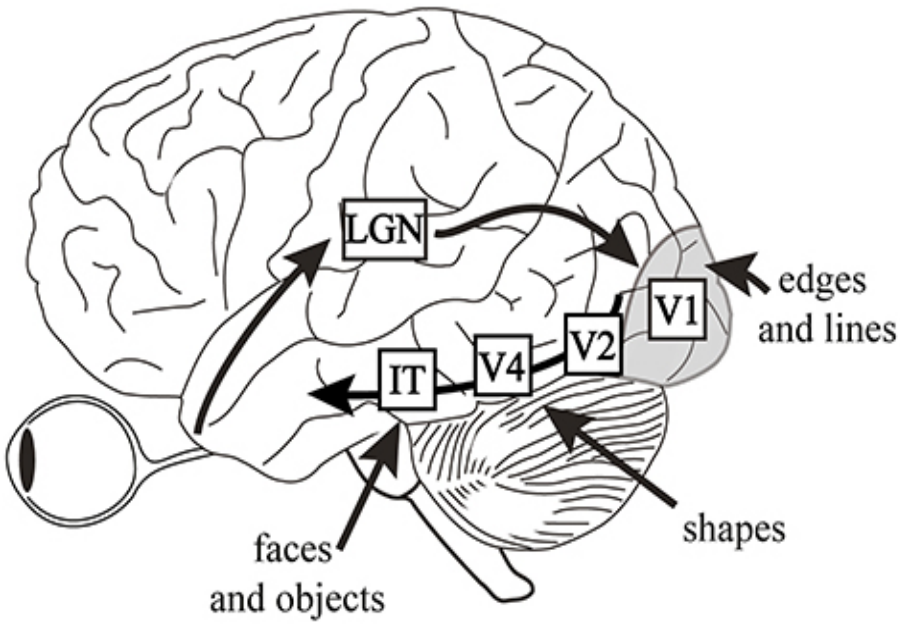
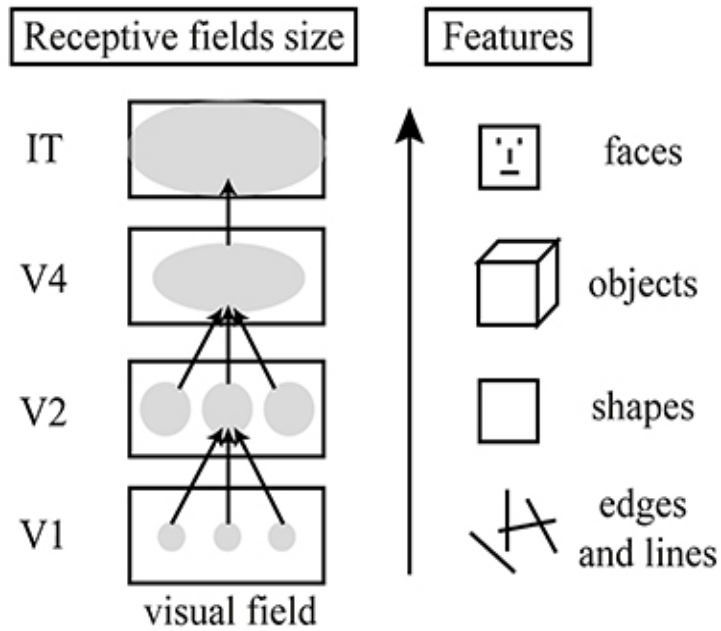


Adversarial Examples

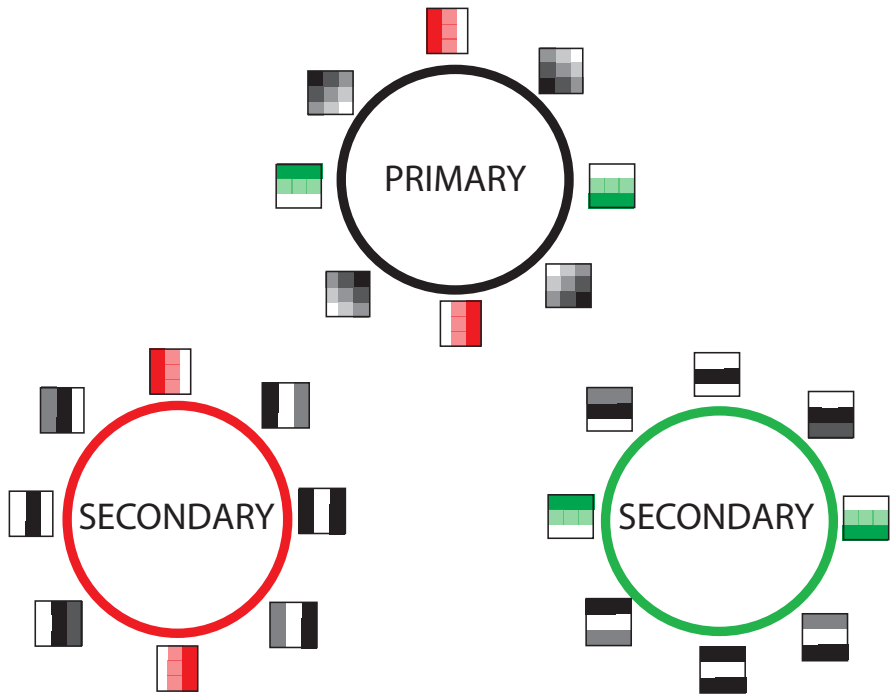


Do CNN's Work Like Mammalian Visual Pathway?





DoCNN's Reflect Natural Image Statistics?



Creating Data Sets from CNN's

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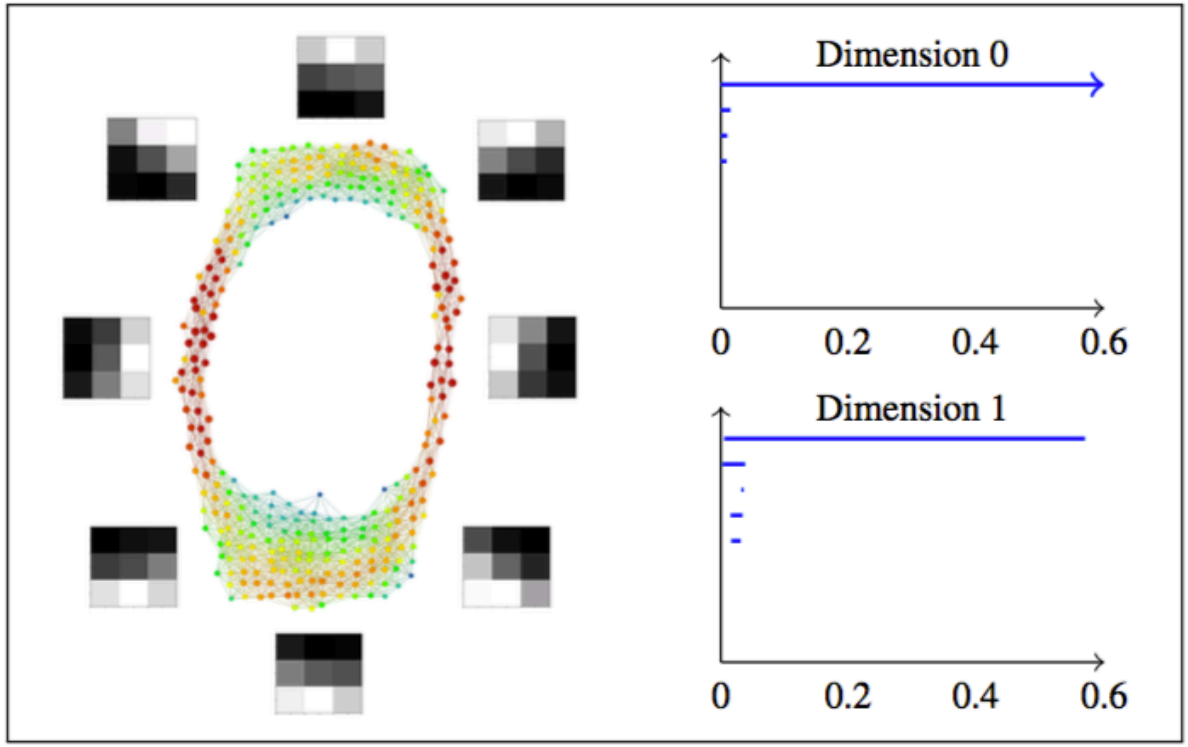
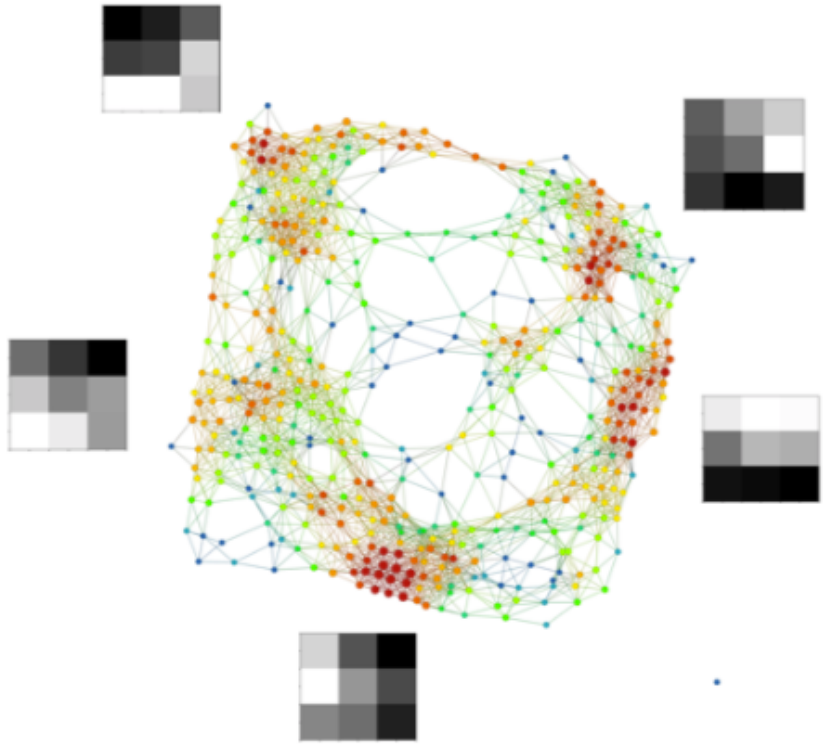
- Apply same methodology as in Lecture 4

Findings: MNIST



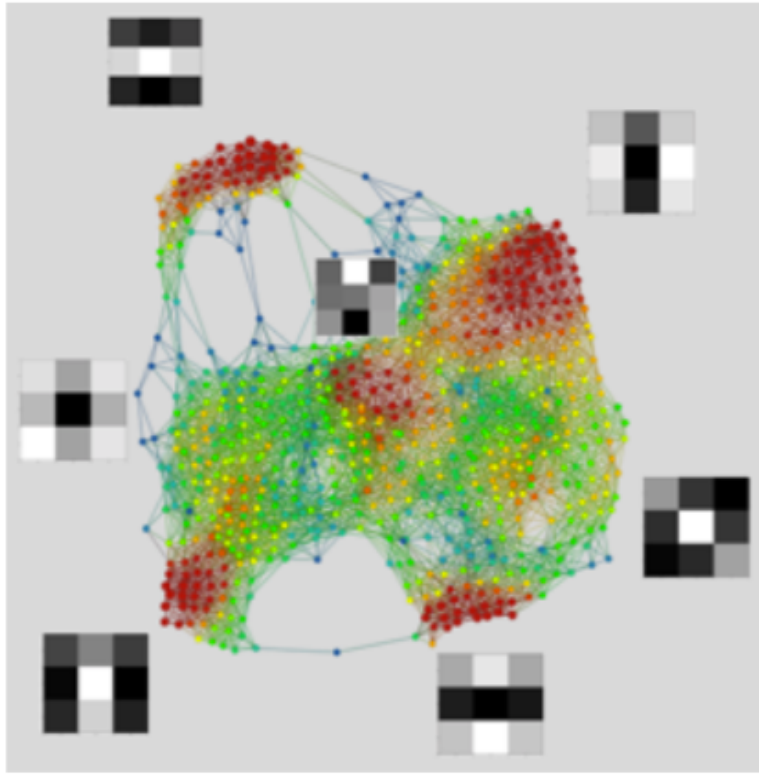
MNIST for layer 1 of a depth two net, non-localized estimator

Findings: MNIST



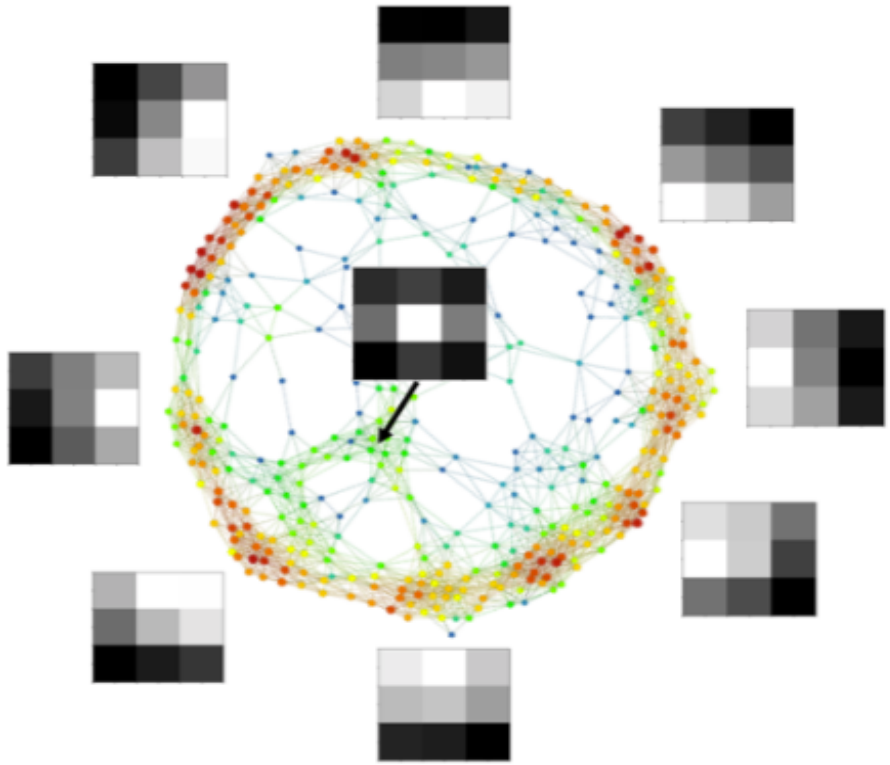
Mnist for layer 1 of a depth two net, localized estimator

Findings: Cifar10



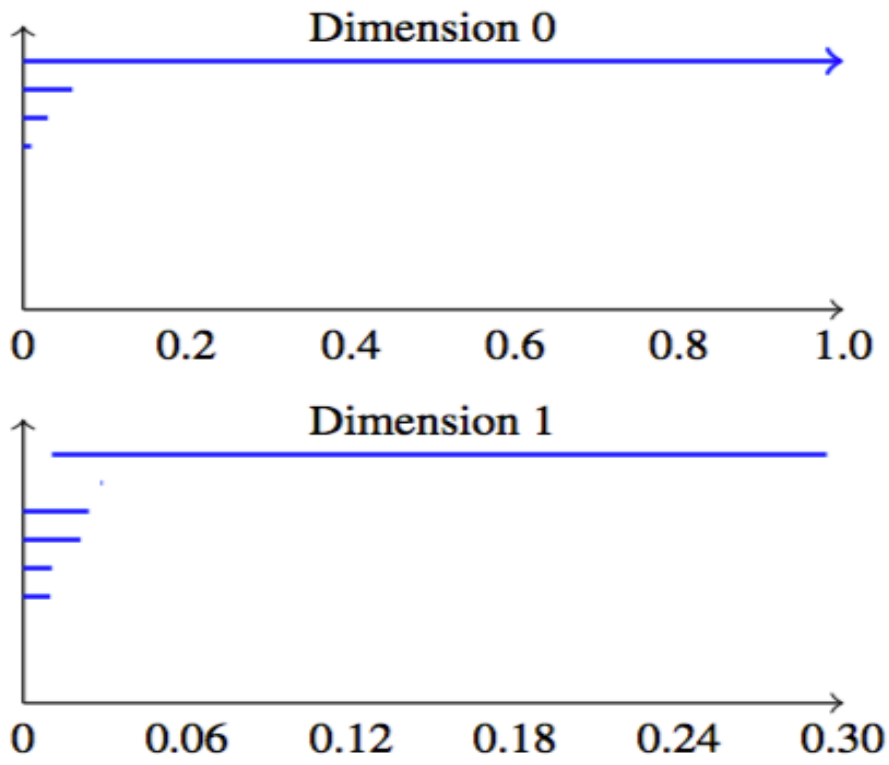
Layer 1 of depth two net, reduced to gray scale

Findings: Cifar10



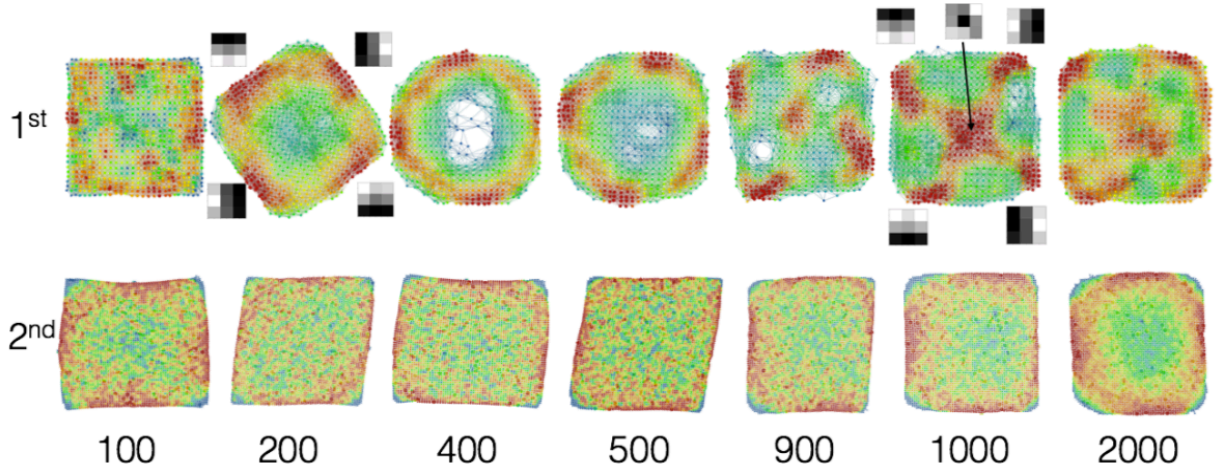
Second layer, reduced to gray scale

Findings: Cifar10



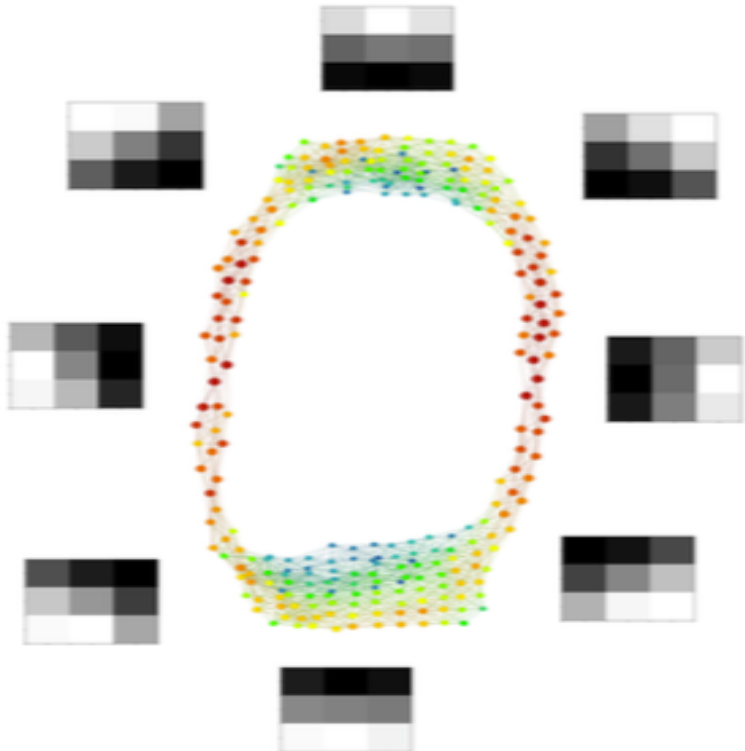
1D barcode for tightly thresholded data set of for 2nd layers , reduced to gray scale

Findings: Cifar10



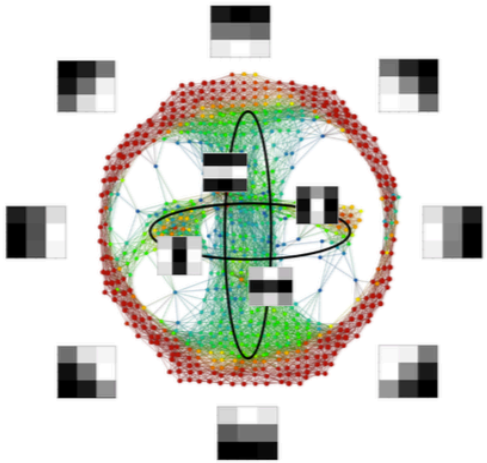
Mapper representations over the number of iterations, tightly density thresholded, gray scale reduced

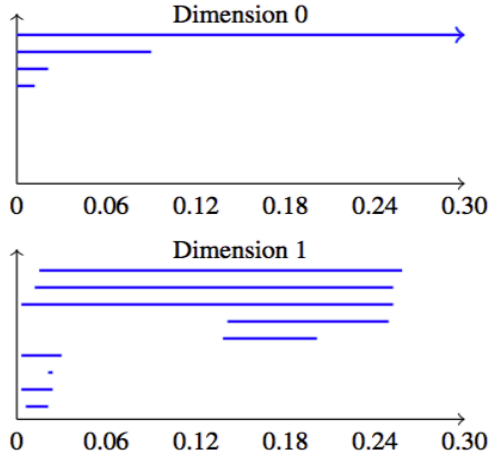
Findings: Cifar10



1st layer, Coarse density thresholding, color retained

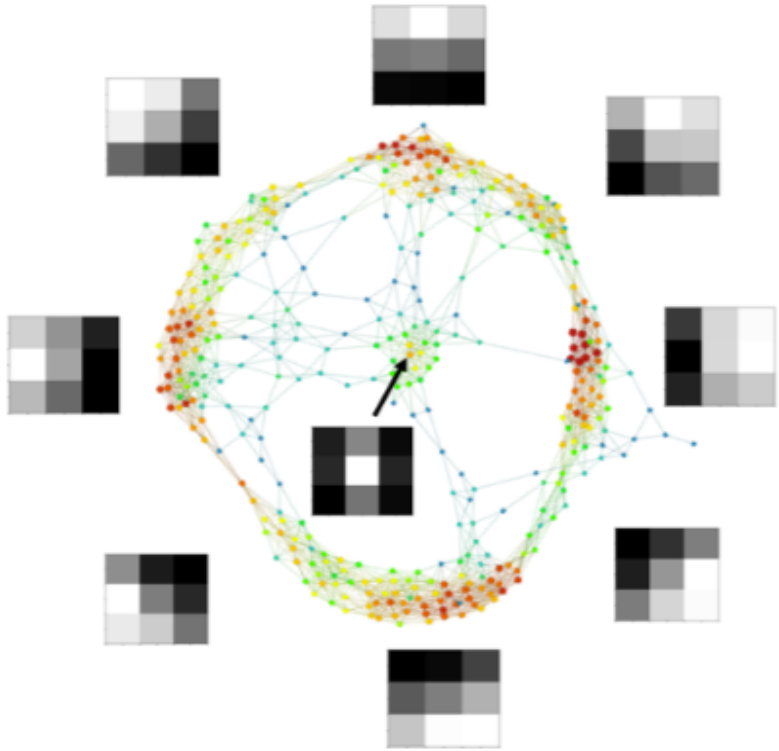
Findings: Cifar10





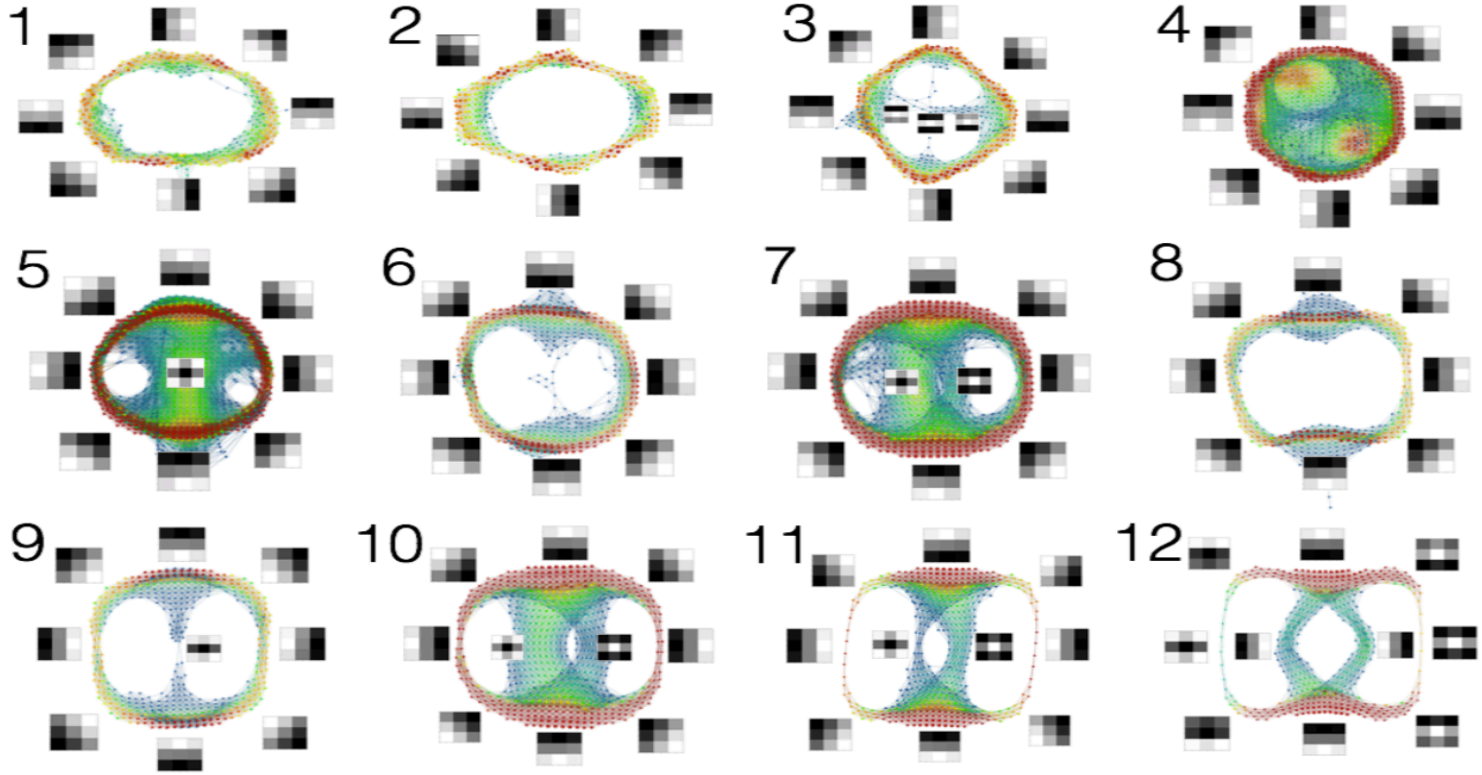
1st layer, looser density thresholding, more localized density estimator, color retained

Findings: Cifar10



2nd layer, fine density thresholding, color retained

Findings: VGG16



Mapper findings from each of 13 layers, same density thresholding, relatively local estimator

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- Add these to existing "raw" features
- Obtain a 3.5x speedup in training for SVHN data set
- Obtain improved generalization for model computed on MNIST applied to SVHN

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- Are there other geometries that might be useful?
- We can build geometries on feature spaces using the primary circle or Klein bottle
- Mapper construction gives compressed geometric structures on the set of features of a general data set

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- Graph of a function is an example
- There is a category $C^{\mathcal{R}}$ whose objects are sets and where the morphisms from X to Y are the relations in $X \times Y$.
- An *abstract feed forward system of depth n* is a functor from the category

$$\underline{n} = 1 \longrightarrow 2 \longrightarrow \cdots \qquad \cdots \longrightarrow n$$

to $c^{\mathcal{R}}$

Building Architectures: Construct Computation Graph from Feed Forward Systems

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- No edges from v to w unless there is an i so that $v \in F(i)$ and $w \in F(i+1)$.

Building Architectures: Important Relations

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- If $f : X \to Y$ is a function, then the graph $\{(x, f(x)|x \in X$ is a relation from X to Y
- If Γ_1 and Γ_2 are two mapper models of the same data set, then there is a naturally defined relation $R(\Gamma_1, \Gamma_2)$ from V_{Γ_1} to V_{Γ_2} given by $(v, w) \in R(\Gamma_1, \Gamma_2)$ if and only if he collections corresponding to v and w have a data point in common

Buillding Architectures

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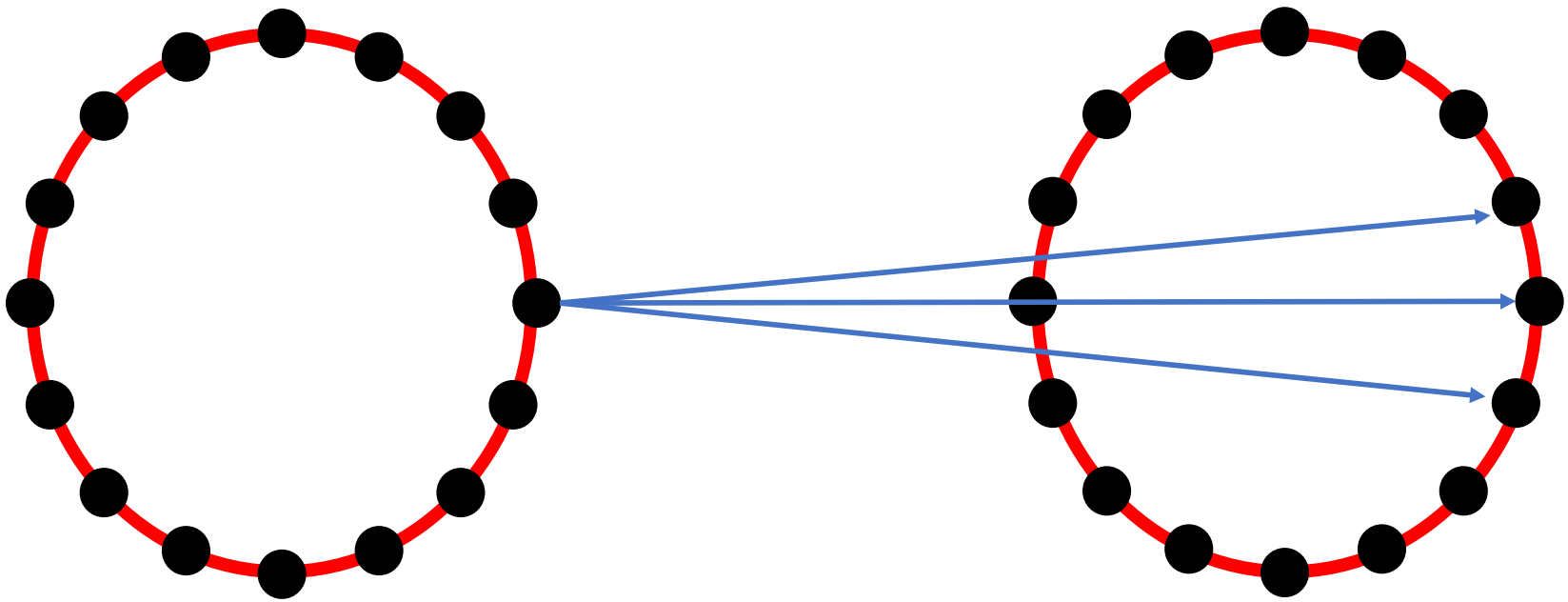
- Bespoke geometries obtained by taking mapper models for the column spaces of data matrices

Building Architectures



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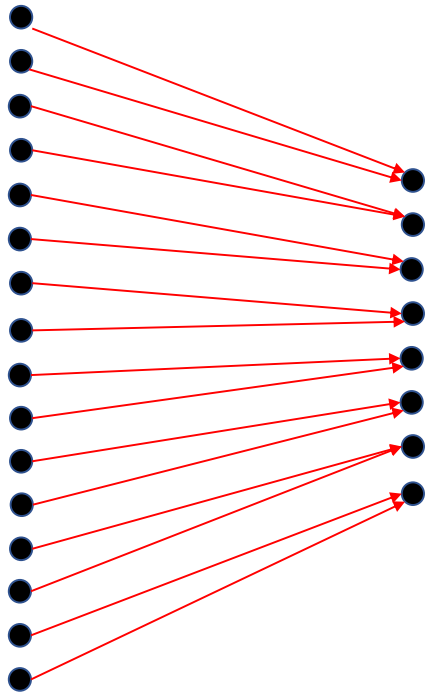

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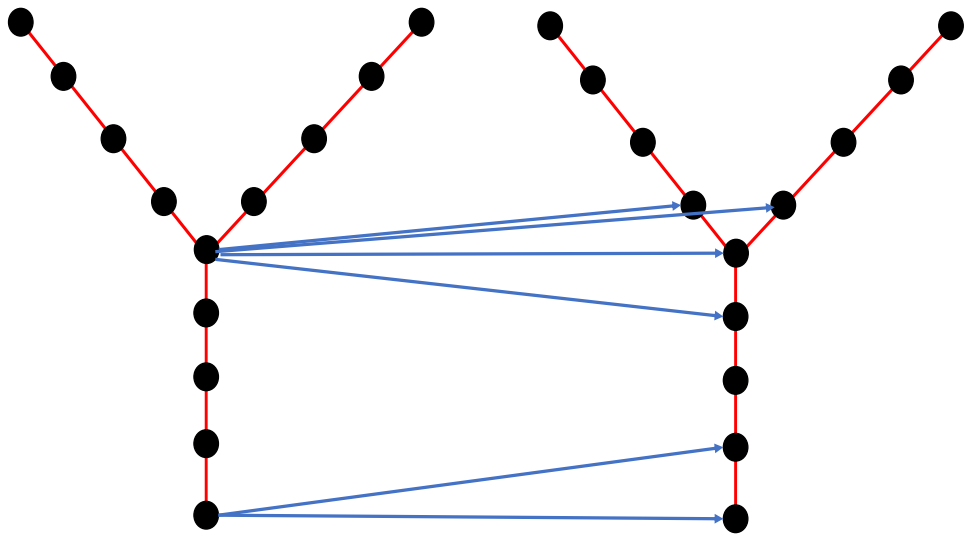

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- Sphere occurs in 3D voxelized images. Icosahedron is a symmetric discretization